Rocket motion (assessment exercise)

A rocket has a mass of 2000 kg of which 1500 kg is fuel. It develops a thrust of $4000Q$ [N], where Q is the fuel mass flow rate in kg/s. It is lifting vertically off the Earth, whose radius R is 6370 km. The gravitational acceleration is inversely proportional to the square of the distance from the centre of the Earth. The atmospheric drag F is given approximately by the equation

$$F(\text{N}) = 0.1\rho v|v|$$

where v is the velocity in m/s (use abs(v) in Visual Basic for the absolute value $|v|$), and the air density ρ (kg/m^3) is given approximately by the equation, $\rho = 1.2e^{-\left(y/7000\right)}$

where y is the height in m.

The fuel mass flow rate varies according to the equation:

$$Q = \begin{cases} \dfrac{Q_0}{2}\left(1+\cos\dfrac{\pi t}{\tau}\right) & \text{for } t < \tau \\ 0 & \text{for } t \geq \tau \end{cases} \qquad \text{where } \tau = \frac{3000}{Q_0}$$

and hence the rocket mass varies according to the equation

$$M = \begin{cases} 2000 - \dfrac{Q_0}{2}\left(t + \dfrac{\tau}{\pi}\sin\dfrac{\pi t}{\tau}\right) & \text{for } t < \tau \\ 500 & \text{for } t \geq \tau \end{cases}$$

(a) Using simple Euler integration, construct a spreadsheet to predict the velocity and altitude over the period from launch to return to earth.

Write a Visual Basic function to calculate the acceleration.

The function will need to start and finish something like this:

```
Function Accel(Altitude, Velocity, Time, Q0)
      Tau = 3000/Q0
      If Time < Tau Then
          :
      Else
          :
      End If
      :
      Accel = ...
End Function
```

Write a Visual Basic macro to implement the Euler integration calculations. The macro should call the acceleration function and write columns for time, velocity, altitude and acceleration in the spreadsheet. Each row should contain formulae for one timestep. Use a *While* loop that stops when the rocket reaches the ground or when an excessive number of calculations has been performed. This may be of the form

```
i = 1
While y >= 0 And i < 10000
    i = i + 1
    a = Accel(y, v, t, q0)
    etcetera
Wend
```

Plot graphs of acceleration, velocity and altitude for an initial fuel flow Q_0 of 20 kg/s.

(b) Tabulate or plot on your spreadsheet the maximum altitude for a range of time steps. Hence determine what timestep is needed for the error in the maximum altitude to be approximately 0.1%.

(c) Modify the spreadsheet or macro to use 4th order Runge-Kutta integration.

Method:

Using Runge-Kutta, for the n^{th} timestep

$$k_{1n} = v_n \qquad g_{1n} = a(y_n, v_n, t_n)$$

$$k_{2n} = v_n + \tfrac{1}{2} h g_{1n} \qquad g_{2n} = a(y_n + \tfrac{1}{2} h k_{1n}, k_{2n}, t_n + \tfrac{1}{2} h)$$

$$k_{3n} = v_n + \tfrac{1}{2} h g_{2n} \qquad g_{3n} = a(y_n + \tfrac{1}{2} h k_{2n}, k_{3n}, t_n + \tfrac{1}{2} h)$$

$$k_{4n} = v_n + h g_{3n} \qquad g_{4n} = a(y_n + h k_{3n}, k_{4n}, t_n + h)$$

$$y_{n+1} = y_n + \frac{h}{6}(k_{1n} + 2k_{2n} + 2k_{3n} + k_{4n}) \qquad v_{n+1} = v_n + \frac{h}{6}(g_{1n} + 2g_{2n} + 2g_{3n} + g_{4n})$$

where y is the altitude, v the velocity, t the time, h the timestep and $a(y, v, t)$ the function for acceleration. $k_{1\text{-}4}$ and $g_{1\text{-}4}$ are intermediate values representing the instantaneous velocity and acceleration at different points in the integration step.

(d) Repeat part (b) for the Runge-Kutta method.

(e) Using <u>either</u> Euler or Runge-Kutta and a suitable timestep, tabulate and plot the maximum altitude for a range of fuel flow rates Q_0. Determine the approximate fuel flow rate that gives the greatest altitude.

<u>Assessment</u>:

By the Deadline, you need to submit a brief report. Include the Visual Basic macro and function(s) as an Appendix. The report (excluding Visual Basic code) should be no more than 1000 words. Graphs should be included and discussed. In particular, explain the features of the acceleration vs time graph. Compare the efficiency of the Euler and Runge-Kutta methods.